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On the static deformation of a bow

B. W. KOOI and J. A. SPARENBERG

Department of Mathematics, University of Groningen, Groningen, The Netherlands

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SUMMARY

The storage of deformation energy in a bow with or without recurve is considered. Some numerical examples are discussed. For a simple bow it is shown that theoretically a shooting efficiency of hundred percent is possible.

1. Introduction

The bow and arrow have been invented by mankind already in prehistoric times. During many millenaries it was its most effective long-range weapon and hunting device. Nowadays it is used in archery, a sport practised by many people all over the world.

A bow can store energy as deformation energy in its elastic arms or limbs. Its special feature is that this energy, delivered by the relatively slow human body, can be quickly released to a light arrow in a very effective way. Probably essential for the effectiveness of the transformation of the deformation energy into kinetic energy of the arrow is the string, as light and inextensible as possible, which couples bow and arrow.

The main object of this paper is to discuss the statics of the bow. It will be represented by an infinitely thin elastic line endowed with bending stiffness which is a function of a length parameter along this line. In the unbraced situation, which is the situation of the bow without string, the elastic line can be curved in the 'opposite' direction. It turns out that this curvature, called recurve, is important with respect to the way in which the deformation energy can be stored. When drawing a bow, in general the force exerted by the archer on the string will increase. So in order to keep a bow in fully drawn position the maximum force, called the weight of the bow, must be exerted by the archer while he aims at the target. Hence one of the objectives for more relaxed shooting is that this force is not too large while still a sufficient amount of deformation energy is stored in the bow. A properly chosen recurve is one of the possibilities to achieve this. It will be shown that by such a recurve it is even possible that the drawing force can decrease in the neighbourhood of maximum draw. Such a phenomenon is well known in the non-linear theory of elasticity.

We will not discuss here the 'compound' bow, invented about fifty years ago by a physicist named Claude Lapp [1]. This bow uses, in order to cause the above-mentioned effect of the decreasing drawing force, pulleys with excentric bearings at the end of the elastic limbs.

Much research has been carried out already on the bow and arrow. For a general background we refer to the article of Klopsteg [2], where many aspects of bow and arrow are thoroughly discussed from a physical point of view. Other papers are for instance [3] and [4], where by making simplifying assumptions calculations of the stored energy have been carried out. In this paper we use the theory of elastica with large deformations as discussed for instance by Frisch-Faÿ [5]. Because nowadays computers are available the non-linear deformation of our model can be calculated without further simplification. It turns out that it can happen, although for not too realistic bows, that there is more than one solution to the problem.

In calculating properties of bows it is the intention to obtain an insight in what makes a bow a good bow, in this paper from the static point of view only. Besides by a number of parameters, length of the bow, ultimate drawing force and some others, the static behaviour of a bow is determined by two functions, namely its shape without string and its distribution of bending stiffness. These functions have to be chosen in one way or another. This means that there is a large measure of freedom which is not so easy to catalogue. It is not the aim of this paper to give a full account of possibilities, however, in the section on numerical results some trends are shown. In a following paper we hope to return to this subject in a more exhaustive way. We remark that when the dynamics of a bow is considered even a third function, the mass distribution, comes into play.

We have also applied our theory to two ancient bows. One is an Asiatic bow of the 14th century and is described in [6]. The other one is much older and is possibly constructed ± 3500 years ago [7].

It should be remarked that in general it is not possible that all the deformation energy stored statically in the bow can be transferred, during the dynamic process of shooting, as kinetic energy to the arrow. This depends on the way in which the kinetic energy of the arms or limbs can be recovered. It is shown in the Appendix, for a simple model of a bow, when the mass of the string can be neglected and when it is inextensible, that all the deformation energy stored in this bow can be transformed into kinetic energy of the arrow. Hence no kinetic energy is left behind in the arms.

2. Formulation of the problem

We will consider bows which are symmetric or nearly symmetric with respect to some line; in the latter case we treat them approximately as being symmetric. The bow is placed in a Cartesian coordinate system $(\bar{x},\bar{y})$, the line of symmetry coinciding with the $\bar{x}$-axis. Its midpoint coincides with the origin 0. The upper half is drawn in Figure 2.1.

We assume the bow to be inextensible and of total length $2\bar{L}$. In our theory it will be represented by an elastic line of zero thickness, along which we have a length coordinate $\bar{s}$ measured from 0, hence $0 \leq \bar{s} \leq \bar{L}$. This elastic line is endowed with bending stiffness $\bar{W}(\bar{s})$.

In Figure 2.1 (a) the unbraced situation (without string) is drawn. The geometry of the bow is described by the local angle $\theta_0(\bar{s})$ between the elastic line and the $\bar{y}$-axis, where $\theta_0(\bar{s})$ is a given function of $\bar{s}$. Because the bow possesses recurve it is predominantly curved to the left.

In Figure 2.1 (b) the bow is braced by applying a string of total length $2\bar{l}$ $(\bar{l} < \bar{L})$, which also is assumed to be inextensible. In the braced position no force in the $\bar{x}$-direction is exerted on

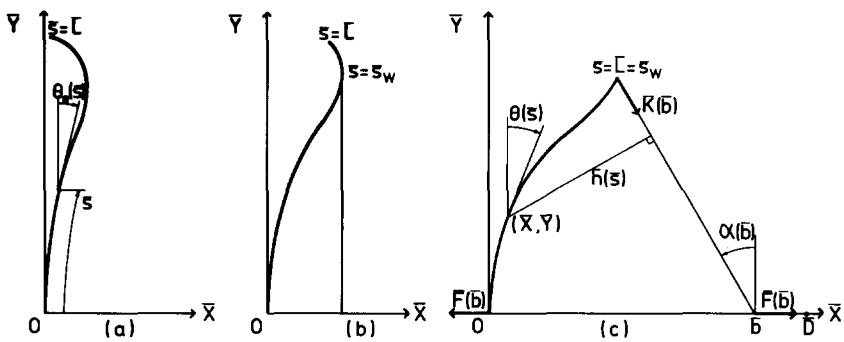


Figure 2.1. Three situations of a bow: (a) unbraced, (b) braced and (c) partly drawn.

the string which intersects the $\bar{x}$-axis with an angle of 90°. It is possible for a bow with recurve as is drawn in Figure 2.1 that a value $\bar{s} = \bar{s}_w < \bar{L}$ exists such that for values of $\bar{s}$ with $\bar{s}_w \leq \bar{s} \leq \bar{L}$ the string lies along the bow. We assume that in that case there is no friction between bow and string. The string at $\bar{s} = \bar{s}_w$ has to be tangent to the bow of which the curvature for $\bar{s}_w \leq \bar{s} \leq \bar{L}$ is the same as the curvature in the unbraced situation. However, it is also possible that for a bow with less or without recurve the string starts from the tip, then $\bar{s}_w = \bar{L}$ and the string can make a non-zero angle with the tangent to the bow at the tip (Figure 4.1). Instead of the length of the string the brace height or 'fistmele' $|\overline{OH}|$ can be used as a basic quantity of the braced position.

In Figure 2.1 (c) the bow is pulled by force $\bar{F}(\bar{b})$ into a partly drawn position where the middle of the string has the $\bar{x}$-coordinate $\bar{b}$. Also in this situation the string can still lie partly along the bow for values of $\bar{s}$ with $\bar{s}_w(\bar{b}) \leq \bar{s} \leq \bar{L}$ and the same considerations hold as were given for the region of contact in the braced situation. To each bow belongs a value $\bar{b} = |\overline{OD}|$ for which it is called fully drawn. The force $\bar{F}(|\overline{OD}|)$ is called the 'weight' of the bow and the distance $|\overline{OD}|$ is its 'draw'.

In our theory we have to consider only the upper half of the bow, clamped at 0. The Bernoulli-Euler equation, which we assume to be valid for the elastic line, reads

$$\bar{M}(\bar{s}) = \bar{W}(\bar{s}) \left(\frac{d\theta}{d\bar{s}} - \frac{d\theta_0}{d\bar{s}} \right), \quad 0 \leq \bar{s} \leq \bar{L}. \tag{2.1}$$

Besides (2.1) we have two geometric equations

$$\frac{d\bar{y}}{d\bar{s}} = \cos\theta, \quad \frac{d\bar{x}}{d\bar{s}} = \sin\theta, \quad 0 \leq \bar{s} \leq \bar{L}. \tag{2.2}$$

The moment $\bar{M}(\bar{s})$ is caused by the tension force $\bar{K}(\bar{b})$ in the string; we find

$$\bar{M}(\bar{s}) = \bar{K}(\bar{b})\bar{h}(\bar{s}) = \bar{K}(\bar{b}) \{\bar{b}\cos\alpha - \bar{x}(\bar{s})\cos\alpha - \bar{y}(\bar{s})\sin\alpha\}, \tag{2.3}$$
$$0 \leq \bar{s} \leq \bar{s}_w,$$

where $\bar{h}(\bar{s})$ (Figure 2.1(c)) is the length of the perpendicular from the point $(\bar{x}(\bar{s}), \bar{y}(\bar{s}))$ to the string and $\alpha(\bar{b})$ is the angle between the string and the $\bar{y}$-axis, reckoned positive in the indicated direction. There are three boundary conditions at $\bar{s} = 0$, namely

$$\theta(0) = \theta_0(0), \quad \bar{x}(0) = \bar{y}(0) = 0. \tag{2.4}$$

Besides we have a geometrical condition with respect to the length of the string. In our model the thickness of the elastic line is assumed to be zero, hence the length of the parts of bow and string which are in contact with each other are equal and we find

$$\{\bar{b} - \bar{x}(\bar{s}_w)\}^2 + \{\bar{y}(\bar{s}_w)\}^2 = \{\bar{l} - (\bar{L} - \bar{s}_w)\}^2. \tag{2.5}$$

When $\bar{b}$ is prescribed the equations (2.1), (2.2) and (2.3) together with the conditions (2.4) and (2.5) are sufficient to determine the situation of the bow, hence also the unknown functions $\theta(\bar{s}), \bar{x}(\bar{s}), \bar{y}(\bar{s})$ and $\bar{M}(\bar{s})$ and the unknown constants $\bar{s}_w(\bar{b}), \bar{K}(\bar{b})$ and $\alpha(\bar{b})$.

It is clear that

$$\bar{M}(\bar{s}) = 0, \quad \bar{s}_w \leq \bar{s} \leq \bar{L}, \tag{2.6}$$

hence it follows from (2.1) that for the region of contact of string and bow, the bow has kept its curvature of the unbraced situation as has been mentioned previously. Thus

$$\theta(\bar{s}) = \theta_0(\bar{s}) + (\theta(\bar{s}_w) - \theta_0(\bar{s}_w)), \quad \bar{s}_w \leq \bar{s} \leq \bar{L}. \tag{2.7}$$

We want to calculate the force $\bar{F}(\bar{b})$ (Figure 2.1 (c)) from which follows the energy $\bar{A}$ stored in the bow when it is brought from the braced position $\bar{b} = |\overline{OH}|$ into the fully drawn position $\bar{b} = |\overline{OD}|$. We have

$$\bar{A} = \int_{|\overline{OH}|}^{|\overline{OD}|} \bar{F}(\bar{b})\, d\bar{b}. \tag{2.8}$$

This amount of energy must be equal to the difference between the deformation energy of the bow in the fully drawn position and the deformation energy in the braced position. Hence we have another representation of $\bar{A}$

$$\bar{A} = \left[\int_0^{\bar{L}} \bar{W}(\bar{s})\,(\theta'(\bar{s}) - \theta_0'(\bar{s}))^2\, d\bar{s} \right]_{\bar{b} = |\overline{OH}|}^{\bar{b} = |\overline{OD}|}, \tag{2.9}$$

which can be used to check the computations.

We now introduce dimensionless quantities by

$$(\bar{x}, \bar{y}, \bar{s}, \bar{L}, \bar{l}, \bar{b}) = (x, y, s, L, l, b) \cdot |\overline{OD}|, \; \bar{M} = M \cdot |\overline{OD}| \cdot \bar{F}(|\overline{OD}|), \; \bar{K} = K \cdot \bar{F}(|\overline{OD}|),$$
$$\bar{W} = W \cdot |\overline{OD}|^2 \cdot F(|\overline{OD}|), \; A = A \cdot |\overline{OD}| \cdot \bar{F}(|\overline{OD}|). \tag{2.10}$$

In (2.10) we have used the still unknown force $\overline{F}(|\overline{OD}|)$ to obtain dimensionless quantities, however, this sometimes makes it more simple to compare numerical results for several types of bows.

Also we introduce the angle

$$\varphi = \theta - \theta_0, \tag{2.11}$$

then after combining (2.1) and (2.3) the relevant equations become

$$W \frac{d\varphi}{ds} = K\{(b - x)\cos\alpha - y \sin\alpha\}\ , \qquad 0 \le s \le s_w, \tag{2.12}$$

$$\varphi(s) = \varphi(s_w) \qquad\qquad , \qquad s_w \le s \le L, \tag{2.13}$$

$$\frac{dx}{ds} = \sin(\varphi + \theta_0), \quad \frac{dy}{ds} = \cos(\varphi + \theta_0), \quad 0 \le s \le L, \tag{2.14}$$

$$\{b - x(s_w)\}^2 + y^2(s_w) = \{1 - L + s_w\}^2, \tag{2.15}$$

$$\varphi(0) = x(0) = y(0) = 0. \tag{2.16}$$

In the next section a method to solve these equations is discussed.

3. Numerical solution of the equation of equilibrium

In this section we consider some aspects of the numerical method used to solve the equations (2.12)-(2.16). We take for b a fixed value

$$\frac{|\overline{OH}|}{|\overline{OD}|} \le b \le 1. \tag{3.1}$$

When b passes through this range the bow changes from its braced position to its fully drawn position. First we assume the bow to be partly or fully drawn, hence not to be in the braced position. The length $2l$ of the string is prescribed.

The unknown force K exerted on the bow by the string passes through the point $(b,0)$ and makes an unknown angle α with the y-axis. We make some choice $\widetilde{K}$ and $\widetilde{\alpha}$ for the values of K and α, and solve the equations (2.12) and (2.14) starting at $s = 0$ where we satisfy the initial conditions (2.16). We assume the functions $W(s)$ and $\theta_0(s)$ to be continuous and $W(s) \ge \epsilon > 0$. Then it is not difficult to show that the solution exists and is unique. A Runge-Kutta method is used to obtain this solution.

There are two possibilities which can occur. First, when continuing the solution of (2.12) and (2.14) for a suitable choice of $\widetilde{K}$ and $\widetilde{\alpha}$, we reach a point A with a value of $s = \widetilde{s}_w < L$ for which

$$\varphi(\tilde{s}_w) = -\tilde{\alpha} - \theta_0(\tilde{s}_w), \tag{3.2}$$

in words, a value of s for which the tangent at the bow is parallel with the chosen direction $\tilde{\alpha}$ of the force $\tilde{K}$. After this the undeformed part AT (Figure 3.1), is added to complete the 'bow', hence

$$\varphi(s) = -\tilde{\alpha} - \theta_0(\tilde{s}_w), \quad \tilde{s}_w \leq s \leq L. \tag{3.3}$$

Second, there is no $\tilde{s}_w < L$ that satisfies (3.2), the solution is continued until $s = L$, then the point A coincides with the tip T of the bow.

So we have found a deflected position OAT of the bow which in fact is caused by connecting to the bow, at $s = \tilde{s}_w$ in the first case or at $s = L$ in the second one, a rigid bar AB perpendicular to the direction $\tilde{\alpha}$, at the end of which acts the force $\tilde{K}$. This is illustrated in Figure 3.1 for the first case.

The force $\tilde{K}$ and the angle $\tilde{\alpha}$ have to be determined such that $|A-B| = 0$ and the 'distance' between the point $(b,0)$ and the tip T measured from A to T along the bow equals l. These two conditions are written as

$$f_1(\tilde{K}, \tilde{\alpha}) \stackrel{\text{def}}{=} \{x(\tilde{s}_w) - b\} \cos\tilde{\alpha} + y(\tilde{s}_w) \sin\tilde{\alpha} = 0, \tag{3.4}$$

and

$$f_2(\tilde{K}, \tilde{\alpha}) \stackrel{\text{def}}{=} y(\tilde{s}_w) - (l - L + \tilde{s}_w) \cos\tilde{\alpha} = 0, \tag{3.5}$$

respectively. The problem is now to solve numerically these two non-linear equations with respect to $\tilde{K}$ and $\tilde{\alpha}$.

For the solution of (3.4) and (3.5) a Newtonian method is chosen. Starting points in the α, K plane for this method have to be close enough to a zero of both f_1 and f_2 to ensure convergence. To obtain starting points we could compute the values of f_1 and f_2 in all nodal points of a grid placed over a suitably chosen region G of the (α,K) plane, where the zeroes are expected. This however would be rather time consuming.

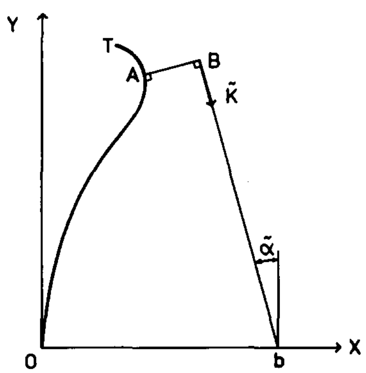


Figure 3.1. The bow deflected by a force $\tilde{K}$, making an angle $\tilde{\alpha}$ with the y-axis, at the arm AB.

Another method is developed in which we move step by step, for instance, along the line $f_2(\alpha,K) = 0$. After each step we check for a change of sign of $f_1(\alpha,K)$. Such a change of sign gives an approximation of a zero of both f_1 and f_2. This procedure has been realized as follows.

For a given value b we take α 'too small', for instance $\alpha = 0$, and using a properly chosen step size we increase K, starting at $K = 0$ and keep $\alpha = 0$. Hence we move along the boundary of the region G, which is a rectangle as drawn in Figure 3.2. Calculating the values of f_2 along this boundary a zero of f_2 can be located approximately by its change of sign in between two succeeding grid points. In Figure 3.2 this point is in between P and Q. By linear interpolation a better approximation ($\alpha_1 = 0, K_1$) for the zero of f_2 is found and $f_1(\alpha_1, K_1)$ is calculated. Now the values of f_2 at R and S are calculated. When there is a change of sign between Q and R, R and S or S and P we know through which side the line $f_2 = 0$ leaves the rectangle $PQRS$. A linear interpolation again gives a better approximation (α_2, K_2) for a zero of f_2 and $f_1(\alpha_2, K_2)$ is calculated. When $f_1(\alpha_1, K_1)$ and $f_1(\alpha_2, K_2)$ have different signs these points are chosen as starting points for the Newtonian method. When there is no change in sign of f_1 we have to start with the adjacent rectangle, of which one side contains the last found approximation for a zero of f_2. This procedure is repeated until we reach the boundary of region G again.

It is assumed that the functions f_1 and f_2 behave sufficiently smooth with respect to the size of the grid placed at the region G. This causes no trouble in practice.

In this way possibly a number of zeroes of the equations (3.4) and (3.5) can be found. Each of these correspond to an equilibrium situation of the bow, while the midpoint of the string has the coordinates $(b,0)$. Not all of these equilibrium situations need to be stable.

We now discuss the braced position which corresponds to $b = |\overline{OH}| \cdot |\overline{OD}|^{-1}$ in (3.1). This value of b, called the brace height or fistmele, is a basic quantity for the adjustment of a bow. We know that $\alpha = 0$ hence we use only equation (3.4) in order to determine the unknown force K. This is done by increasing K stepwise from zero and checking for a change in sign of f_1. By iteration K can be determined with sufficient accuracy. Then equation (3.5) gives us the half length l of the string.

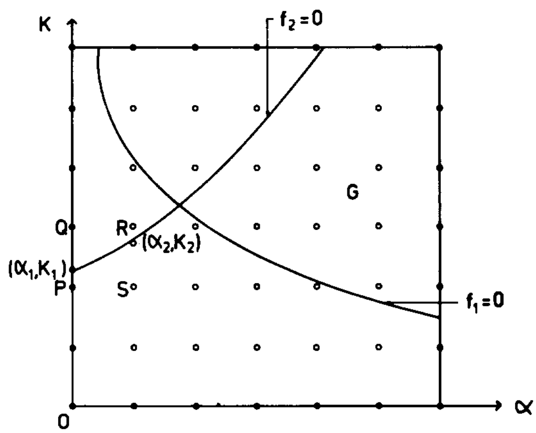


Figure 3.2. Determination of the zero's of $f_1 = 0$ and $f_2 = 0$.

The braced position of the bow can also be determined by prescribing the half length l of the string. Then equations (3.4) and (3.5) can be considered as equations for the two unknowns b and K, again $\alpha = 0$. A procedure analogous to the one prescribed in the first part of this section can be used to satisfy both (3.4) and (3.5). Small changes, however, in the length $2l$ of the string can cause rather large variations of the fistmele b. Because this is in general the more important quantity, the first method to calculate the braced position of the bow is recommended.

To check our program we have compared solutions obtained by it with solutions obtained by other methods. We can take the bending stiffness W constant and the bow straight in unbraced situation, $\theta_0(s) \equiv 0$. For given values of l and b our program yields the values of K and α. Now we can use the theory of the largely deflected cantilever described in [5] to compute the strain energy due to bending caused by the forced defined by K and α. The elliptic integrals needed for this computation are obtained by linear interpolation of values in the tables given in [8]. The results agreed very well and differed only by an amount of 0.1%.

Another check has been made by using, in the case of a bow without recurve, the finite-element program *MARC* of the *MARC* Analysis Research Corporation. Also these results agreed with ours: a comparison of the drawing force $F(b)$ showed discrepancies of only 0.5%.

4. Some numerical results

As we mentioned already, it is important for a bow to possess a sufficient amount of deformation energy at full draw, kept in check by a not too large ultimate force or weight. The measure in which the bow meets this demand can be described to a certain extent by a dimensionless number q, called the static quality coefficient. Suppose we have an amount of deformation energy $\bar{A}$ in the bow in the situation of full draw $\bar{b} = |\overline{OD}|$ and the force is $\bar{F}$, then

$$q = \frac{\bar{A}}{|\overline{OD}| \cdot \bar{F}} = A, \tag{4.1}$$

where the second equality follows from (2.10). The dimensionless deformation energy A depends on a number of parameters and functions,

$$q = A(L, W(s), \theta_0(s), OH \text{ or } l), \quad 0 \leq s \leq L. \tag{4.2}$$

This number q is also a measure for the concavity of the function $F = F(b)$. When we compare two bows with the same value of $|\overline{OD}|$, one with a larger q than the other, the first bow is from the static point of view the best because it can store more deformation energy 'per unit of weight'. Sometimes another definition of q is given by replacing $|\overline{OD}|$ in (4.1) by $|\overline{OH}|$. Then, however, when $|\overline{OH}|$ is changed the just mentioned property is no longer valid. It is clear that q cannot give a decisive answer to questions about shooting efficiency. In the case of a real bow the lengths $|\overline{OH}|$ and $|\overline{OD}|$ have to be measured from a reasonably chosen elastic line representing the bow, to the midpoint of the string.

One of our objectives is to get insight into the dependence of q on the quantities denoted in

(4.2). To this end we start with the bow described in [3] and change in a more or less systematic way its parameters and functions.

Some bows possess a nearly rigid central section of which the grip forms part of, its length being denoted by $2\bar{L}_0$. From the ends of this section extend the elastic limbs each of length $\bar{L}_1$, the half length of the bow being $\bar{L} = \bar{L}_0 + \bar{L}_1$. For the grip, hence for $0 \leq \bar{s} \leq \bar{L}_0$, we put $\bar{W}(\bar{s}) = \infty$.

The units we use are cm (= 0.3937 inch) and kg force (= 2.205 lbs). Because in the literature characteristic lengths are often given in inches by 'simple' numbers, for instance $\bar{L}_0 = 4$ inch, $|\overline{OD}| = 28$ inch, these lengths expressed in cm sometimes suggest an accuracy which is not intended. The same holds for lbs and kg. In the following we do not mention anymore the dimension of a quantity. It is tacitly understood that a length is expressed in cm, a force in kg, a bending stiffness in kgcm^2, an energy in kgcm and an angle in radians.

The bow (H bow) discussed in [3] by Hickman has the following characteristics.

$$\bar{L} = 91.4,\ \bar{L}_0 = 10.2,\ \theta_0(\bar{s}) \equiv 0,\ \overline{OH} = 15.2. \tag{4.3}$$

The bending stiffness distribution for $\bar{s} > \bar{L}_0$ is a linear function

$$\bar{W}(\bar{s}) = 1.30.10^5\ \frac{(\bar{L} - \bar{s})}{\bar{L}}\ ,\ \bar{L}_0 \leq \bar{s} \leq \bar{L}. \tag{4.4}$$

For future reference we mention $\bar{W}(\bar{L}_0) = 1.15.10^5$. For the draw of the bow we have chosen $|\overline{OD}| = 71.1$ which is slightly different from the value used in [3]. However when we compare Hickman's theory with this one, his results are corrected for this difference.

It follows from (4.4) that $\bar{W}(\bar{L}) = 0$. Because we use the Euler-Bernoulli equations this is not a difficulty from the theoretical point of view, because the limit of the curvature of the elastic line for $\bar{s} \to \bar{L}$ remains finite. However, in order to avoid computational complications we put

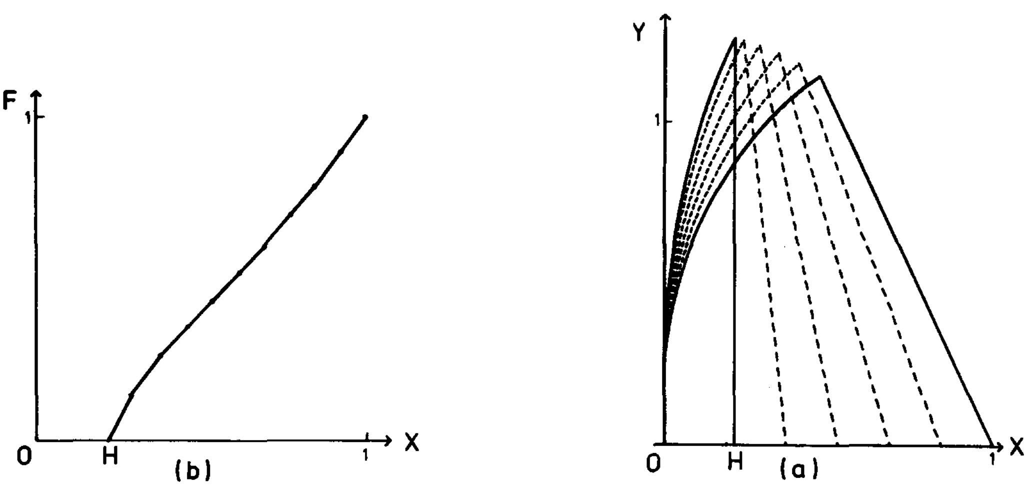


Figure 4.1. (a) Some shapes of the dimensionless deformations of the H bow. (b) Dimensionless force-draw curve of the H bow.

$$\bar{W}(\bar{s}) = 7.69, \tag{4.5}$$

whenever in (4.4) the values of $\bar{W}(\bar{s})$ become smaller than 7.69. This interpretation has to be given also to other bending stiffness distributions which occur later on.

A number of times we consider the consequences of a change of one or more characteristic quantities of the H bow. This means that only these quantities are varied while the other ones are the same as those given above. We remark that our results for the weight of a bow $\bar{F}(|\overline{OD}|)$ and for the deformation energy $\bar{A}$ are linearly dependent on λ when we replace $\bar{W}(\bar{s})$ by $\lambda\bar{W}(\bar{s})$. Hence it is easy to adjust the weight of a described bow to a desired value by multiplying $\bar{W}(\bar{s})$ by a suitable λ. The quality factor q is independent of λ.

In Figure 4.1 (a) we have drawn a number of dimensionless deformations of this bow upto its fully drawn position and in Figure 4.1 (b) its dimensionless force-draw curve both calculated by this theory. When curves given by Hickman are made dimensionless there is an excellent agreement with Figure 4.1 although the numbers with dimension show some difference. Numerical results of Hickman's theory and of this one are given in Table 1.

In Table 2 we show the influence of a change of the length of the grip $\bar{L}_0$ and the brace height $|\overline{OD}|$ of a H bow. It is seen that the largest value of q occurs for the smallest grip and smallest brace height and the smallest value of q for the largest grip and largest brace height, however, this difference is not very spectacular.

In Table 3 we give the influence of a change of the length $\bar{L}$ of a H bow. It follows that the weight of the bow increases strongly when the bow becomes shorter while there is, as in Table 2, only a weak influence on the quality factor q.

TABLE 1.

Comparison between Hickman's theory and this theory.

	Hickman	this theory
$\bar{F}(\lvert\overline{OD}\rvert)$	15.1	15.5
$\bar{A}$	444	450
q	0.414	0.407

TABLE 2.

Influence of length of grip $\bar{L}_0$ and of brace height $|\overline{OH}|$ on the H bow.

$\bar{L}_0$	5.08			10.2			15.2		
$\overline{OH}$	12.7	15.2	17.8	12.7	15.2	17.8	12.7	15.2	17.8
$\bar{F}(\lvert\overline{OD}\rvert)$	13.9	14.0	14.1	15.4	15.5	15.7	17.3	17.4	17.6
$\bar{A}$	417	408	398	460	450	439	510	500	488
q	0.423	0.411	0.397	0.420	0.407	0.393	0.405	0.403	0.389

Next we discuss the influence of a change of the bending stiffness on the H bow. We choose

$$\bar{W}_n(\bar{s}) = 1.15.10^5 \left(\frac{\bar{L}-\bar{s}}{\bar{L}-\bar{L}_0}\right)^{\beta_n}, \quad \bar{L}_0 \leq \bar{s} \leq \bar{L}, \quad n = 1,2,3,4, \tag{4.6}$$

with $\beta_1 = 0$, $\beta_2 = \frac{1}{2}$, $\beta_3 = 1$, $\beta_4 = 2$. We refer with respect to $\bar{W}(\bar{L}) = 0$, to (4.5) and the remark belonging to it. The bending stiffness $\bar{W}_3(\bar{s})$ is equal to $\bar{W}(\bar{s})$ from (4.4). With increasing values of n the relative flexibility of the tip becomes larger. The results are given in Table 4. We find that an increase of the stiffness of the tip causes some increase of q, and that the bending stiffness distribution has only a modest favourable influence on q for n changing from 3 to 1.

We now consider the influence of two recurve shapes denoted by $\theta_{0,1}(\bar{s})$ and $\theta_{0,2}(\bar{s})$, on the H bow. The first one is very simple, $\theta_{0,1}(\bar{s}) = -0.12$, the second one $\theta_{0,2}(\bar{s})$ is given by the unbraced shape of the bow in the (x,y) plane in Figure 4.2, where for reference also $\theta_{0,1}(\bar{s})$ is drawn. For each of these bows we have used $\bar{W}_2(\bar{s})$ as well as $\bar{W}_3(\bar{s})$ as bending stiffness distribution. The results are given in Table 5. For reference we also give in this table the straight bow $\theta_0(\bar{s}) = 0$, which already is given in Table 4 under the headings $\bar{W}_2(\bar{s})$ and $\bar{W}_3(\bar{s})$. It is seen that both recurves have statically a favourable influence on the bow because the coefficient q is in both cases larger than q belonging to $\theta_0(\bar{s}) \equiv 0$. The recurve $\theta_{0,2}(\bar{s})$ has the highest values of q. The best one of these $q = 0{,}575$ (which is a rather large value) occurs for $\bar{W}_3(\bar{s})$ which has a more flexible tip than $\bar{W}_2(\bar{s})$. It is remarkable that this is opposite to that of recurves $\theta_0(\bar{s})$ and $\theta_{0,1}(\bar{s})$, where the highest q occurs for $\bar{W}_2(\bar{s})$.

Next we consider two bows B_1 and B_2 also with recurve of which the unbraced situation,

TABLE 3.

Influence of the length $\bar{L}$ on the H bow.

$\bar{L}$	81.3	86.4	91.4	96.5	102
$\bar{F}(\lvert\overline{OD}\rvert)$	25.0	19.5	15.5	12.6	10.4
$\bar{A}$	698	555	450	370	309
q	0.393	0.400	0.407	0.413	0.417

TABLE 4.

Influence of the bending stiffness $\bar{W}(\bar{s})$ on the H bow.

	$\bar{W}_1(\bar{S})$	$\bar{W}_2(\bar{S})$	$\bar{W}_3(\bar{S})$	$\bar{W}_4(\bar{S})$
$\bar{F}(\lvert\overline{OD}\rvert)$	23.6	19.6	15.5	9.02
$\bar{A}$	701	576	450	229
q	0.417	0.414	0.407	0.356

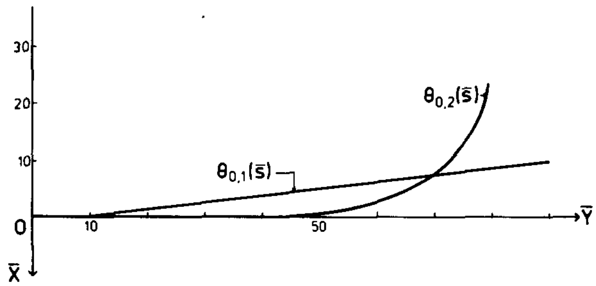


Figure 4.2. The two types of recurve $\theta_{0,1}(\bar{s})$ and $\theta_{0,2}(\bar{s})$ considered in Table 5.

TABLE 5.

Influence of recurve on the H bow for two bending stiffness distributions.

		$\theta_0(\bar{s}) \equiv 0$	$\theta_{0,1}(\bar{s})$	$\theta_{0,2}(\bar{s})$
$\bar{W}_2(\bar{s})$	$\bar{F}(\lvert\overline{OD}\rvert)$	19.6	25.0	38.4
	$\bar{A}$	576	776	1510
	q	0.414	0.437	0.554
$\bar{W}_3(\bar{s})$	$\bar{F}(\lvert\overline{OD}\rvert)$	15.5	20.1	29.2
	$\bar{A}$	450	607	1200
	q	0.407	0.424	0.575

however, differs from those of the bows we considered up to now. The bow B_1 drawn in Figure 4.3 (a) is a normal modern recurve bow for target shooting.

The difference with bows considered before is that the elastic limb starts at the end $s = \bar{L}_0$ of the rigid section in the direction of the archer. Its measured bending stiffness distribution is given in Figure 4.3 (b). The bow B_2 has an excessive recurve (Figure 4.3 (c)). Its bending stiffness varies linearly from the rigid section to the tip (Figure 4.3 (d)). In Table 6 we give the parameters of these bows and the calculated quantities $\bar{F}(|\overline{OD}|), \bar{A}$ and q.

It is remarkable that the static quality coefficient q of B_2 is very high with respect to all the bows we have considered. The reason is that the main part of its force draw curve is strongly convex as can be seen from the dimensionless force draw curve of Figure 4.4, where also the curves of bow B_1 and of the bow B_3 denoted by $(\theta_{0,2}(\bar{s}), \bar{W}_3(\bar{s}))$ in Table 5 are given. This shape of force-draw curve (bow B_2) resembles the force draw curve of the compound bow mentioned in the introduction, here however no pulleys are needed. In Figure 4.5 we have drawn the dimensionless deformation curves of B_1 and B_2.

We emphasize that it is not clear that B_2 will be a good bow for shooting because our considerations are only based on statics. However, it seems worthwhile to investigate the dynamical behaviour of this bow which will depend also on the choice of the mass distribution of the elastic limbs.

The next bow resembles an Asiatic bow ([6], plate 18). It has a rather strong recurve. We have tried to guess a bending stiffness so that its calculated braced and fully drawn position resemble the photographs given in [6]. Opposite to the bows discussed up to now this bow has a rigid tip which is strengthened by a ridge. A difficulty is that this bow does not show too clearly a line of symmetry; it even is said that the upper limb is the shooting limb which

TABLE 6.

Two recurve bows B_1 and B_2

	$\bar{L}$	$\bar{L}_0$	$\overline{OH}$	$\overline{OD}$	$\bar{F}(\lvert\overline{OD}\rvert)$	$\bar{A}$	q
B_1	84.2	28.4	17.4	62.5	13.6	362	0.426
B_2	81.9	14.8	15.2	71.1	13.6	854	0.883

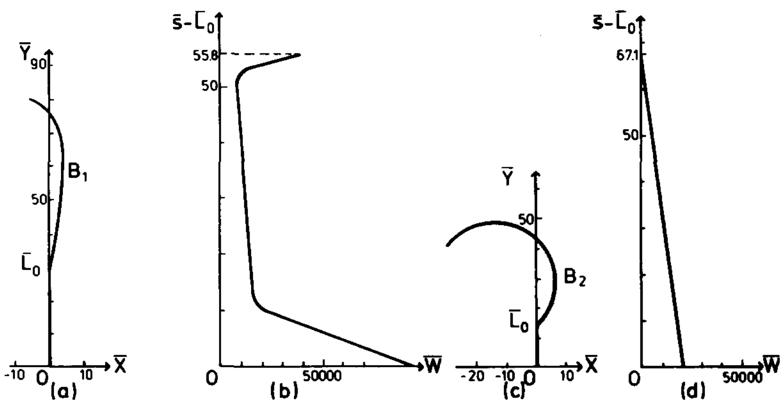


Figure 4.3. (a), (b) a modern recurve bow B_1, $\theta_0(\bar{s})$ and $\bar{W}(\bar{s})$, (c), (d) a bow with strong recurve B_2, $\theta_0(\bar{s})$ and $\bar{W}(\bar{s})$.

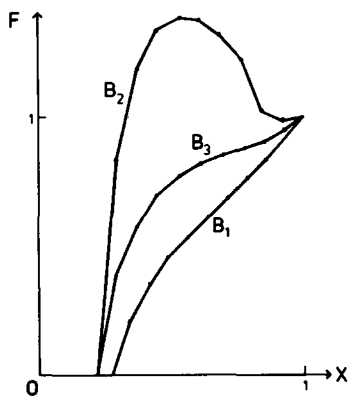


Figure 4.4. The dimensionless force draw curves of B_1, B_2 and the bow B_3 of Table 4 ($\theta_{0,2}(\bar{s})$, $\bar{W}_3(\bar{s})$).

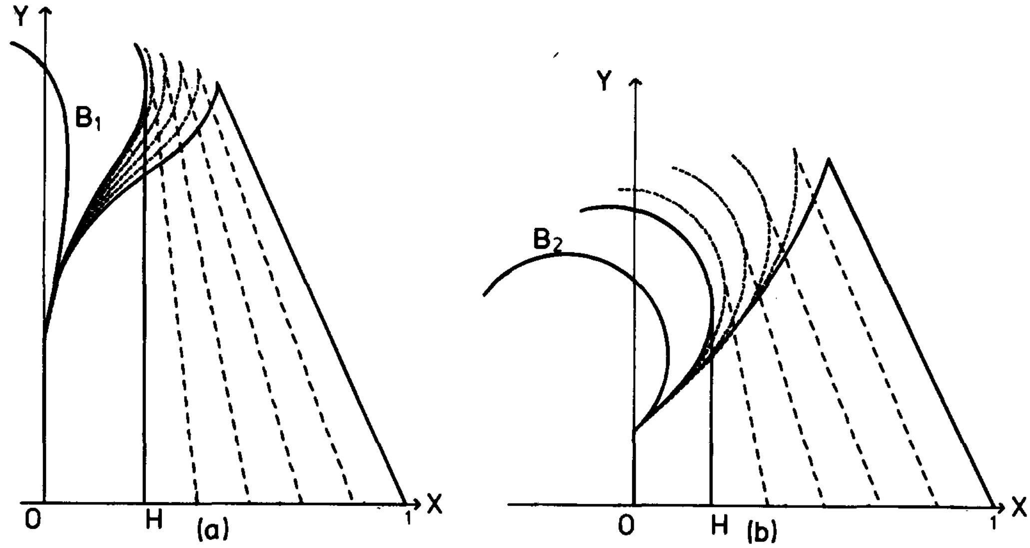


Figure 4.5. Dimensionless deformation curves of the bows B_1 and B_2.

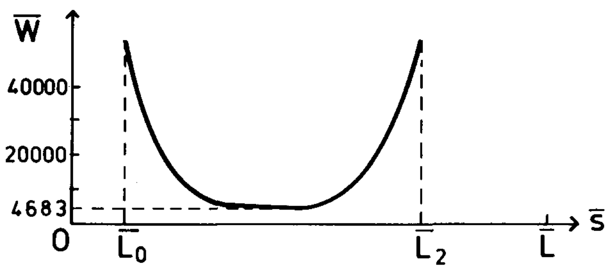


Figure 4.6. Chosen bending stiffness of 'Asiatic bow'.

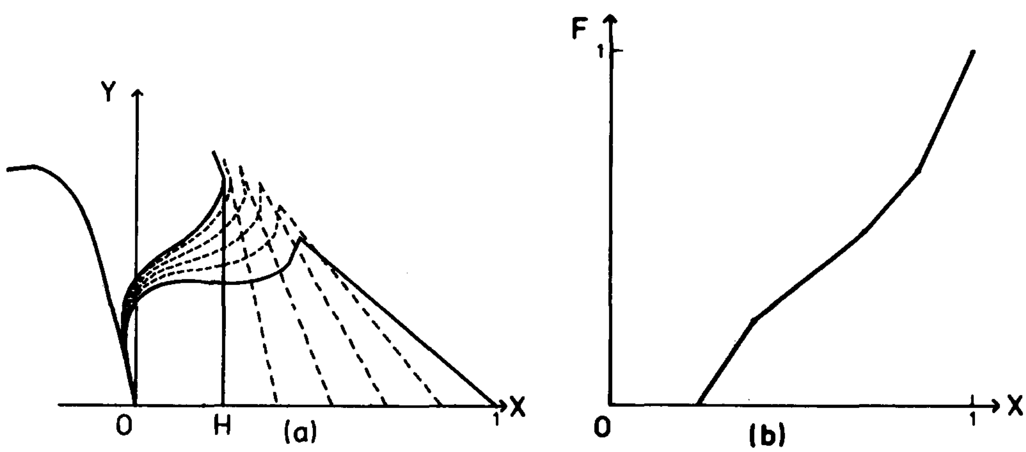


Figure 4.7. Dimensionless deformation curves and force-draw curve of 'Asiatic bow'.

'accounts for most of the shooting'. In Figure 4.6 we give its chosen bending stiffness distribution. For $0 \leq \bar{s} \leq \bar{L}_0 = 6.24$ and for $46.8 = \bar{L}_2 \leq \bar{s} \leq \bar{L} = 63.5$ we take $\bar{W} = \infty$. The braceheight $|\overline{OH}| = 18.4$ and the draw $|\overline{OD}| = 76.2$. Figure 4.7 gives the dimensionless deformation curves and force draw curve. From Figure 4.7 (a) we have also an impression of its unbraced shape. Calculated quantities are $\bar{F}(|\overline{OD}|) = 22,7, \bar{A} = 586$ and $q = 0.339$. Hence its static quality factor q is rather low.

We also consider a bow found in the neighbourhood of Vrees which is described by Beckhoff [7]. The quantities measured or guessed, given in that paper, are $\bar{L} = 83.8, \bar{L}_0 = 0, |\overline{OH}| = 17$, $|\overline{OD}| = 70$, $\theta_0(s) = 0$ and $\bar{W}(\bar{s})$ is given in Table 7. The weight and deformation energy calculated in [7] and by this theory are given in Table 8. The reason for the discrepancies between the two calculations is possibly that Berkhoff used a linearized theory and other approximations. It is remarkable that q is the same in both theories.

At last we give an example of the possibility of more than one braced situation of a bow.

TABLE 7.

Bending stiffness of the bow of Vrees.

$\bar{s}$	0	15.2	27.5	35.8	44	64.5	69.8	83.8
$\bar{W}(\bar{s}) \cdot 10^{-4}$	35.4	27.0	23.8	21.7	16,6	5.31	3.24	1.26

TABLE 8.

Comparison of results of [7] and this theory.

	Beckhoff	this theory
$\bar{F}(\lvert\overline{OD}\rvert)$	27.2	45.1
$\bar{A}(\lvert\overline{OD}\rvert)$	748	1240
q	0.393	0.393

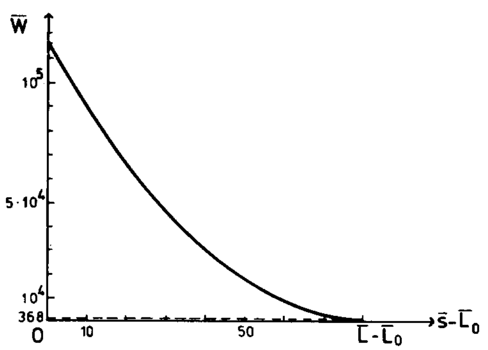
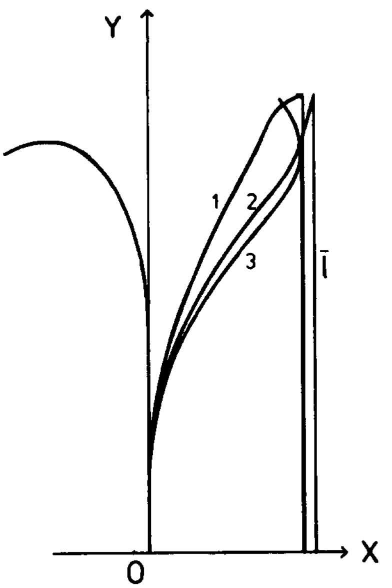


Figure 4.8. Bending stiffness of bow with three braced positions.



	1	2	3
$\bar{A}(\overline{|OH|})$	1090	1120	1080

Figure 4.9. Three possible equilibrium positions.

This phenomenon is liable to happen because our theory is non-linear and it can be expected to occur when the tip of the bow is rather flexible with respect to its central parts. In order to find several situations we prescribe the length $\bar{l}$ of the string instead of the brace height or fistmele. We have chosen $\bar{L} = 90.4$, $\bar{L}_0 = 10.2$, $\bar{l} = 82.9$. Its bending stiffness $\bar{W}(\bar{s})$ is given in Figure 4.8 and $\theta_0(\bar{s})$ follows from Figure 4.9. The three braced positions are drawn in Figure 4.9 and denoted by 1,2 and 3. When we perturb these shapes in a number of ways, it was numerically found that 1 and 3 possibly belong to a local minimum of the deformation energy and 2 belongs to a maximum. In other words it seems that the shapes 1 and 3 are stable and 2 unstable although this has not been proved analytically.

Appendix

A model of a bow with 100% shooting efficiency

Although the main subject of this paper is the static deformation of a bow we will show, as is already announced in the introduction, the essential importance of the string for a good shooting efficiency.

A shooting efficiency of 100% can easily be obtained if the model of the bow is unrealistically simple. Consider a bow of which the elastic limbs and the string are without mass, then it is clear that all the deformation energy is transformed into kinetic energy of the arrow which is assumed to have a non-zero finite mass. The assumption of a string without mass seems accept-

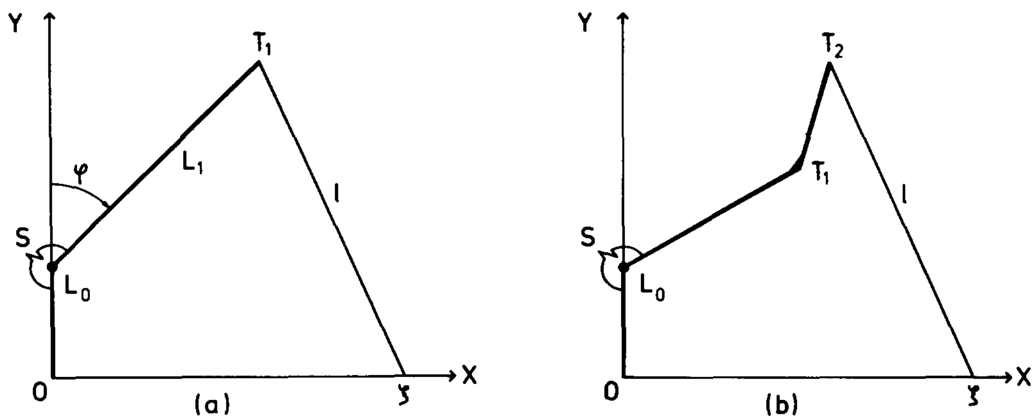


Figure A.1. Two bows each with two elastic hinges and rigid limbs.

able however the assumption of limbs without mass is not at all in correspondence with reality. Therefore we now discuss a more realistic model.

The bow consists of a rigid grip of length $2L_0$ and two rigid limbs of length L_1 (Figure A.1(a)) which are connected each to the grip by means of an elastic hinge (S for the upper limb) of strength $k > 0$. The moment of inertia of the limb with respect to S is J. The string of length $2l$ is inextensible and without mass, the mass of the arrow is $m > 0$. The assertion is that this bow (Figure A.1(a)) converts all the deformation energy of the elastic hinge into kinetic energy of the arrow. From Figure A.1(a) it follows that

$$L_1 \cos\varphi \leq l - L_0 < L_1, \varphi_0 \stackrel{\text{def}}{=} \arccos \frac{(l - L_0)}{L_1} \leq \varphi < \varphi_e \stackrel{\text{def}}{=} \pi - \arccos\left(\frac{L_0}{L_1 + l}\right). \tag{A.1}$$

Also from that figure we find for the x-coordinate ξ of the end of the arrow

$$\xi = L_1 \sin\varphi + \{l^2 - (L_1 \cos\varphi + L_0)^2\}^{1/2}. \tag{A.2}$$

Writing down the equations of motion of limbs and arrow we find after a straight forward analysis:

$$\ddot{\xi}\{J + \frac{m}{2} Q^2(\varphi)\} = JQ'(\varphi)\dot{\varphi}^2 - k(\varphi - \tilde{\varphi})Q(\varphi), \tag{A.3}$$

where

$$Q(\varphi) = [L_1 \cos\varphi + L_1 \frac{(L_1 \cos\varphi + L_0)\sin\varphi}{\{l^2 - (L_1\cos\varphi + L_0)^2\}^{1/2}}] \geq 0, \quad \varphi_0 \leq \varphi < \varphi_e, \tag{A.4}$$

and $\tilde{\varphi} < \varphi_0$ is the angle of zero moment of the elastic hinge. Because it can be shown that $Q'(\varphi) \leq 0$ it follows from (A.3) that

$$\ddot{\xi} \leq 0, \quad \varphi_0 \leq \varphi < \varphi_e. \tag{A.5}$$

An important conclusion results from this equation. During the stretching of the bow (Figure A.1(a)) the arrow keeps its contact with the string which, along straight lines, connects the arrow end to the tips of the limbs.

Next we consider the bow of Figure A.1(b). The only difference between this bow and the previous one is that now the rigid limb ST_2 has an infinitely sharp bend at T_1. During positions as drawn this bow behaves exactly as the one of figure A.1(a), hence $\ddot{\xi} \leq 0$. When, however, $T_2 - T_1$ coincides partly with the string we can describe the process of shooting after that situation, by a bow of which the limb is ST_1 and of which the half length of the string is $(l - |T_1 - T_2|)$. It is easily proved that $\dot{\xi}$ is continuous during this transition and hence also for this bow we have $\ddot{\xi} \leq 0$ for all possible values of φ. This means that *mutatis mutandis*, for this bow the same conclusion (below (A.5)) holds.

Now consider the situation that the string is nearly stretched

$$\varphi_0 \leq \varphi \leq \varphi_0 + \epsilon, \tag{A.6}$$

for a small number $\epsilon > 0$. From (A.5) it follows that the arrow is still in contact with the string. Suppose that for these values of φ the angular velocity of the limb is non zero, hence that a positive number δ exists with

$$0 < \delta \leq -\dot{\varphi}(\varphi). \tag{A.7}$$

Then it follows from (A.2)

$$\lim_{\varphi \to \varphi_0} \dot{\xi}(\varphi) = \lim_{\varphi \to \varphi_0} Q(\varphi)\,\dot{\varphi} = -\infty. \tag{A.8}$$

However this is impossible because then the kinetic energy of the arrow becomes infinite while the deformation energy of the bow is finite. Hence we have

$$\lim_{\varphi \to \varphi_0} \dot{\varphi}(\varphi) = 0. \tag{A.9}$$

This means that theoretically by the action of the inextensible string without mass all the kinetic energy of the rigid limbs is conveyed to the arrow, how large J and how small m may be. This holds for both bows of Figure A.1, it holds analogously for bows with rigid limbs with more sharp bends.

That these models are not too unrealistic follows for the type of Figure A.1(a) from [9] where an analogous device, a catapult, is described. The elastic hinges are made of strongly twisted cables to which rigid limbs are connected. The bow of Figure A.1(b) resembles a Turkish flight bow [6] (page 105). There it is remarked that the ancient bowyers tried to keep the elastic parts of the limbs as short as possible in order to obtain a good shooting efficiency. With other words they tried to realize an elastic hinge in each of the limbs. The purpose of the bend at T_1 in the rigid limb in Figure A.1(b) is to increase the value of q.

By choosing non-linear elastic hinges, which are not difficult to design, it is of course possible to obtain force-draw curves of the type of B_2 of Figure 4.4, hence to obtain a high static quality factor q. When contact between string and arrow remains during the shooting, in other words, when the accelaration of the arrow is non positive also a 100% shooting efficiency can be obtained.

It seems likely that suitably designed bows with more elastic hinges or even continuously distributed elasticity, can also have theoretically an efficiency of 100%. An analytic proof however will be more complicated in that case.

Acknowledgement

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